Remarks on Robin's and Nicolas Inequalities

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Abstract

Nicolas Conjecture is disproved. The Robin Conjecture follows.

To V. I. Arnold, the greatest mathematician of all times.

I believe this to be false. There is no evidence whatever for it (unless one counts that it is always nice when any function has only real roots). One should not believe things for which there is no evidence. In the spirit of this anthology I should also record my feelings that there is no imaginable reason why it should be true.
J. E. Littlewood on Riemnann's Conjecture

... We are all in our own eyes a failure: after all,
we haven't proved Fermat's Last Theorem,
nor Riemann's Conjecture.
Mary L. Cartwright

1 Introduction

The Nicolas Conjecture [Nic 1983] states that

$$\frac{\mathcal{N}_k}{\varphi(\mathcal{N}_k)} > e^{\gamma} loglog \mathcal{N}_k, \quad k \geq 1, \tag{0}$$

where:

$$\mathcal{N}_k = \prod_{i=1}^{k} p_i,$$

p_i is the prime number $\#i$, φ is the Euler phi-function, and $\gamma = 0.57...$ is the Euler constant. For more details, see the beautiful paper [CLM 2006], where it was proven that

$$\prod_{i=1}^{k} (p_i + 1)/p_i < e^{\gamma} loglog \mathcal{N}_k^2, \ k > 4,$$

in contrast to (1):

$$\prod_{i=1}^{k} p_i/(p_i-1) > e^{\gamma} loglog\mathcal{N}_k,\ \ k \geq 1.$$

Crucially, Nicolas proved that if his Conjecture is *not* true then the inequality (1) is both true and untrue infinitely often. Thus, it's enough to establish it for $k >> 1$, i.e. for k large enough.

The Robin's inequality, equivalent to Nicolas one, is:

$$\frac{\sigma(n)}{n} < e^{\gamma} loglogn, \quad n \geq 5041, \tag{1}$$

where $\sigma(n) = \sigma_1(n) = \sum_{d|n} d$ is the sum of divisors of n. [Rob 1984].

As is clear from the table of exceptions ≤ 5040 in [CLM 2006], (1) can be replaced by:

$$\frac{\sigma(n)}{n} < e^{\gamma} loglogn, \quad \omega(n) > 4, \tag{2}$$

where $\omega(n)$ is the number of distinct prime divisors of n.

CLMS also mention that it's enough to consider only those n which are in Hardy-Ramanujan form: if $\omega(n) = m$,

$$n = p_1^{e_1} ... p_m^{e_m}, \tag{3}$$

then

$$e_1 \geq e_2 \geq ... \geq e_m.$$

Still, the inequality (1) has arbitrary many parameters: the e_i's. This is not conducive to a proof. The Nicolas conjecture offers better chances.

2 The Method

Thus, in handling the Nicolas inequality, we need

$$logLHS \overset{?}{>} logRHS,$$

where

$$logLHS = \sum_{i=1}^{m} log\left(1 + \frac{1}{p_i - 1}\right), \tag{4}$$

$$logRHS = \gamma + loglog\theta(p_m). \tag{5}$$

Now,

$$\theta(p_m) \le p_m\left(1+\frac{\eta_s}{log^s p_m}\right),\ s=1,2,3\ \Rightarrow$$
$$log\theta(p_m) \le logp_m + \frac{\eta_s}{log^s p_m} = logp_m\left(1+\frac{\eta_s}{log^{s+1}p_m}\right) \Rightarrow$$
$$loglog\theta(p_m) \le loglogp_m + \frac{\eta_s}{log^{s+1}p_m}, \tag{6}$$

and we are going to look, beyond first 2 terms, $\gamma + loglogp_m$, at the series in $\frac{1}{logp_m}$. Thus,

$$logRHS = loglogp_m + \gamma + \frac{\eta_3}{log^4 p_m} + ... \tag{7}$$

For the LHS, we have:

$$\sum_{i=1}^{n} log\left(1+\frac{1}{p_i-1}\right) = \sum_{i=1}^{m}\left\{\left[log\left(1+\frac{1}{p_i-1}\right)-\frac{1}{p_i}\right]+\frac{1}{p_i}\right\} =$$
$$= \sum_{i=1}^{m}\left[log\left(1+\frac{1}{p_i-1}\right)-\frac{1}{p_i}\right] + \sum_{i=1}^{m}\frac{1}{p_i} = \sum_{i-1}^{m}\frac{1}{p_i} + \sum_{i=1}^{\infty} log\left(1+\frac{1}{p_i-1}\right)-\frac{1}{p_i}\Bigg] -$$
$$-\sum_{m+1}^{\infty}\left[log\left(1+\frac{1}{p_i-1}\right)-\frac{1}{p_i}\right]\left(\Big[\text{Fin } 2003\Big], \text{p.}95\right) =$$
$$= \sum_{i=1}^{m}\frac{1}{p_i} + (\gamma - M) - \sum_{m+1}^{\infty} log\left(1+\frac{1}{p_i-1}\right)-\frac{1}{p_i}\Bigg], \tag{8}$$

where ([Fin 2003], p. 95)

$$\sum_{i=1}^{m}\frac{1}{p_i} = loglogp_m + M + 0(1), \tag{9}$$
$$M = 0.261497... \tag{10}$$

Let's dispose now of $\sum_{m+1}^{\infty}$ - term. We have:

$$log\left(1+\frac{1}{p-1}\right) - \frac{1}{p} \sim \frac{1}{p-1} - \frac{1}{p} \sim \frac{1}{i^2 log^2 i}$$

and

$$\int_{m+1}^{\infty}\frac{dx}{x^2 log^2 x} \sim -\frac{1}{xlog^2 x}\Bigg|_{m+1}^{\infty} \sim \frac{1}{mlog^2 m},$$

and we are going to count only m-free terms in comparing the log LHS with the log RHS.

Thus, the first two leading terms: $loglogp_m$ and γ - being equal on both sides, we are comparing

$$\sum_{i=1}^{m} \frac{1}{p_i} - loglogp_m - M \text{ and } 0, \tag{11}$$

at least modulo

$$\frac{1}{log^3 m}. \tag{12}$$

We need the first nonzero term in the log LHS, in (11), being > 0 (or < 0, as the case may be).

3 The Proof

We calculate modulo $1/m^2$ and modulo $1/mlog^4 m$. Our coefficients are polynomial functions in $w = loglogm$.

Lemma 13.

$$log(m+1) = logm + \frac{1}{m}. \tag{14}$$

Proof.

$$log(m+1) = log[m\left(1 + \frac{1}{m}\right)] = logm + log\left(1 + \frac{1}{m}\right) \equiv logm + \frac{1}{m} \quad \blacksquare$$

Lemma 15.

$$loglog(m+1) = loglogm + \frac{1}{mlogm}. \tag{16}$$

Proof.

$$loglog(m+1) = log\left(logm + \frac{1}{m}\right) \equiv log\left[logm\left(1 + \frac{1}{mlogm}\right)\right] =$$

$$= loglogm + \frac{1}{mlogm}. \quad \blacksquare \tag{1}$$

Let $C = C(w), w = loglogm$. Call it C_m.

Lemma 17.

$$\frac{C_m}{log^k m} - \frac{C_{m+1}}{log^k(m+1)} = \frac{kC - C'}{mlog^{k+1} m}. \tag{18}$$

Proof. We have:

$$LHS = \frac{C}{log^k m} - \frac{C + C' \frac{1}{mlogm}}{\left(logm + \frac{1}{m}\right)^k} = \frac{1}{log^k m\left(log^k m + \frac{k}{m} log^{k-1} m\right)} \text{ times :}$$

$$C\left(log^k m + \frac{k}{m} log^{k-1} m\right) - \left(C + C' \frac{1}{mlogm}\right) log^k m =$$

$$= \frac{log^{k-1} m}{m}(kC - C').$$

Altogether,

$$LHS = \frac{1}{mlog^{k+1} m}\ (kC - C'). \ \blacksquare$$

Lemma 19.

$$\frac{1}{p_{m+1}} = \frac{1}{p_m}. \tag{20}$$

Proof. Set

$$p_m = mf(m), \tag{21}$$

where

$$f(m) = logm + (loglogm - 1) + \frac{loglogm - 2}{logn} + ... =$$

$$= logm + \sum_{i \geq 0} \frac{P_i}{log^i m}. \tag{22}$$

Denote $P(m+1) = \tilde{P}$. Then

$$\frac{1}{p_m} - \frac{1}{p_{m+1}} = \frac{1}{p_m p_{m+1}}(p_{m+1} - p_m) \sim \frac{1}{m^2 log^2 m}(p_{m+1} - p_m), \tag{23}$$

and

$$p_{m+1} - p_m = (m+1)f(m+1) - mf(m) =$$

$$= m[f(m+1) - f(m)] + f(m+1) \equiv m[f(m+1) - f(m)],$$

and

$$f(m+1) - f(m) = log(m+1) - logm + \sum_{i \geq 1}\left(\frac{\tilde{P}_i}{log^i(m+1)} - \frac{P_i}{log^i m}\right).$$

But

$$log(m+1) - logm = \frac{1}{m},$$

and by (18),

$$\frac{\tilde{P}_i}{log^i(m+1)} - \frac{P_i}{log^i m} = \frac{P_i' - iP_i}{mlog^{i+1}m}.$$

Together with (23), this proves (20). ■

We are interested in $C = C_1, D = C_2$, such that

$$\sum_{i=1}^{m} \frac{1}{p_i} = loglogp_m + M + \frac{C}{logm} + \frac{D}{log^2 m} + ...$$

We use induction on m to find C and D. We have:

$$\sum_{i=1}^{m} \frac{1}{p_i} + \frac{1}{p_{m+1}} = \frac{1}{p_{m+1}} + loglogp_m + M + \frac{C}{logm} + \frac{D}{log^2 m} =$$

$$= loglogp_{m+1} + M + \frac{\tilde{C}}{log(m+1)} + \frac{\tilde{D}}{log^2(m+1)}, \tag{24}$$

or

$$\left(\frac{C}{logm} - \frac{\tilde{C}}{log(m+1)}\right) + \left(\frac{D}{log^2 m} - \frac{\tilde{D}}{log^2(m+1)}\right) = (loglogp_{m+1} - loglogp_m) - \frac{1}{p_m}. \tag{25}$$

By (18), this is:

$$\frac{C - C'}{mlog^2 m} + \frac{2D - D'}{mlog^3 m} = (loglogp_{m+1} - loglogp_m) - \frac{1}{p_m}. \tag{26}$$

Let's start with the $1/p_m-$ term. We have:

$$-\frac{1}{p_m} = -\frac{1}{mlogm}\left(\frac{1}{1 + \frac{P_0}{logm} + \frac{P_1}{log^2 m}}\right) =$$

$$= -\frac{1}{mlogm}\left(1 - \frac{P_0}{logm} - \frac{P_1}{log^2 m} + \frac{P_0^2}{log^2 m}\right). \tag{27}$$

Lemma 28.

$$logp_{m+1} - logp_m \sim \frac{1}{m}\left(1 + \frac{1}{logm}\right) - \frac{P_0' - P_0}{mlog^2 m}. \tag{29}$$

Proof. We have:

$$LHS = log\frac{p_{m+1}}{p_m} = log\left[\frac{m+1}{m}\frac{f(m+1)}{f(m)}\right] = \frac{1}{m} + log\frac{f(m+1)}{f(m)} =$$

$$= \frac{1}{m} + log\left\{\frac{log(m+1)}{logm}\left[\frac{1+\frac{\tilde{P}_0}{log(m+1)}+\frac{\tilde{P}_1}{log^2(m+1)}+...}{1+\frac{P_0}{logm}+\frac{\tilde{P}_1}{log^2m}+...}\right]\right\} =$$

$$= \frac{1}{m} + \frac{1}{mlogm} + log\left[\frac{\cdots}{\cdots}\right] =$$

$$= \frac{1}{m}\left(1+\frac{1}{logm}\right) + log\left[1+\frac{\tilde{P}_0}{log(m+1)}+\frac{\tilde{P}_1}{log^2(m+1)}+...\right] - log\left[1+\frac{P_0}{logm}+\frac{P_1}{log^2m}+...\right] =$$

$$= \frac{1}{m}\left(1+\frac{1}{logm}\right) + \frac{\tilde{P}_0}{log(m+1)} - \frac{P_0}{logm} \quad \text{[by (18)]} =$$

$$= \frac{1}{m}\left(1+\frac{1}{logm}\right) - \frac{P_0' - P}{mlog^2m}. \quad \blacksquare$$

Lemma 30.

$$\frac{1}{logp_m} = \frac{1}{logm}\left(1-\frac{loglogm}{logm}\right) + O\left(\frac{1}{log^3m}\right). \tag{31}$$

Proof. We have:

$$\frac{1}{logp_m} = \frac{1}{logm+logf(m)} = \frac{1}{logm}\frac{1}{[1+\frac{1}{logm}logf(m)]} =$$

$$= \frac{1}{logm}\left(1-\frac{loglogm}{logm}\right) \quad \blacksquare$$

Lemma 32.

$$loglogp_{m+1} - loglogp_m = \frac{1}{mlogm}\left(1+\frac{1}{logm}\right) + \frac{P_0'-P_0}{mlog^3m} - \frac{loglogm}{mlog^2m}\left(1+\frac{1}{logm}\right). \tag{33}$$

Proof. We have:

$$LHS = log\frac{logp_{m+1}}{logp_m} = log\left(1+\frac{logp_{m+1}-logp_m}{logp_m}\right) =$$

$$= \frac{logp_{m+1}-logp_m}{logp_m} \text{[by (29), (31)]} =$$

$$= \frac{1}{logm}\left(1-\frac{loglogm}{logm}\right)\left\{\frac{1}{m}\left(1+\frac{1}{logm}\right) - \frac{P_0'-P_0}{mlog^2m}\right\} =$$

$$= \frac{1}{mlogm}\left(1+\frac{1}{logm}\right) - \frac{P_0'-P_0}{mlog^3m} - \frac{loglogm}{mlog^2m}\left(1+\frac{1}{logm}\right). \quad \blacksquare$$

Collecting all Lemmas, we rewrite (25) as

$$\frac{C - C'}{mlog^2m} + \frac{2D - D'}{mlog^3m} =$$
$$= \frac{1}{mlogm}\left(1 - \frac{P_0}{logm} + \frac{P_0^2 - P_1}{log^2m}\right) +$$
$$+\frac{1}{mlogm}\left(1 + \frac{1}{logm}\right) - \frac{P_0' - P_0}{mlog^3m} - \frac{loglogm}{mlog^2m}\left(1 + \frac{1}{logm}\right). \tag{34}$$

Thus,

$$\frac{1}{mlogm} - \text{coefficient is } 0;$$
$$\frac{1}{mlog^2m} - \text{coefficient is :}$$
$$P_0 + 1 - loglogm = 0 \text{ because } P_0 = loglogm - 1;$$
$$\frac{1}{log^3m} - \text{coefficient is :}$$
$$-P_0^2 + P_1 - P_0' + P_0 - loglogm < 0 \text{ for } m >> 0. \text{ Thus,}$$
$$LHS < RHS. \tag{35}$$

The Nicolas Conjecture is thus disproved, but the strengthened Robin Inequality, is established:

$$\prod_{i=1}^{m} \frac{p_i}{\varphi(p_i)} < e^{\gamma} loglog\mathcal{N}_m.$$

Notice that Nicolas Inequality is disproved in a way that contradicts the Nicolas result, that if his inequality is wrong, then it's wrong *and right* infinitely often. Note that the equation

$$kF - F' = p(w),$$

where p is a polynomial, has trivially a unique polynomial solution in $w = loglogm$.

4 Acknowledgment 1

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5 Acknowledgment 2

It would be churlish of me not to mention the wonderfully well written and remarkably exciting and stimulating paper [CLMS 2006].

6 Acknowledgment 3

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7 Acknowledgment 4

The dim-witted editors of the Archive had removed the 1^{st} version of the Manuscript "for inappropriate language". When I appealed their thuggish action to Grinspurg, Kuperberg, and Tao, I got not response.

8 Acknowledgment 5

This paper might have never been finished were it not for the vaguely unthreatening letter from Thi-Eve Jao, the Editor-in-Chief of the International Journal of Chinese Mathematics, in which he politely explained that idleness of people like me threatens the livelihood of people like him, and implored me to finish the proof of the Nicolas conjecture as expeditiously as possible, but no later than the opening of the next Congress of the Chinese Communist Gangsters Party. There was nothing else to do but to comply with so exquisitely framed request.

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